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A CONSISTENCY RESULT ON THIN-VERY TALL BOOLEAN ALGEBRAS

BY

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ABSTRACT

It was proved by Baumgartner and Shelah that Con (ZFC) $\rightarrow$ Con (ZFC + "there is a superatomic Boolean algebra of width ω and height ω_2"). In this paper we improve Baumgartner–Shelah's theorem, showing that Con (ZFC) $\rightarrow$ Con (ZFC + "for every $\alpha < \omega_3$ there is a superatomic Boolean algebra of width ω and height α").

0. Introduction

A Boolean algebra B is **superatomic** if every subalgebra of B is atomic. It is a well-known fact that a Boolean algebra B is superatomic iff its Stone space $S(B)$ is scattered. The Cantor–Bendixson process for topological spaces can be transferred to the context of Boolean algebras, obtaining in this way an increasing sequence of ideals. Suppose that B is a Boolean algebra. Then, for every ordinal α we define *the ideal* I_α as follows: $I_0 = \{0\}$; if $\alpha = \beta + 1$, $I_\alpha =$ the ideal generated by $I_\beta \cup \{b \in B: b/I_\beta$ is an atom in $B/I_\beta\}$; and if α is a limit, $I_\alpha = \bigcup\{I_\beta: \beta < \alpha\}$. Then, B is superatomic iff there is an ordinal α such that $B = I_\alpha$.

We define **the height** of a superatomic Boolean algebra B by $\mathrm{ht}(B) =$ the least ordinal α such that B/I_α is finite (which means $B = I_{\alpha+1}$). For every $\alpha < \mathrm{ht}(B)$, we denote by $\mathrm{wd}_\alpha(B)$ the cardinality of the set of atoms of B/I_α,

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and then we define **the width** of B by $\mathrm{wd}(B) = \sup\{\mathrm{wd}_\alpha(B)\colon \alpha < \mathrm{ht}(B)\}$. If κ is an infinite cardinal and η is a nonzero ordinal, we say that B is a (κ,η)-BA, if B is superatomic, $\mathrm{wd}(B) = \kappa$ and $\mathrm{ht}(B) = \eta$. B is **κ-thin-tall**, if B is a (κ,η)-BA for some $\eta \geq \kappa^+$. B is **κ-thin-very tall**, if B is a (κ,η)-BA for some $\eta \geq \kappa^{++}$.

Juhász and Weiss proved in [4] that for every $\alpha < \omega_2$, there is an (ω,α)-BA. It is not known whether there is an (ω_1,ω_2)-BA. Nevertheless, it was shown in [8] that for any specific regular cardinal κ and any specific ordinal $\eta < \kappa^{++}$, the existence of a (κ,η)-BA is consistent with ZFC. Also, Baumgartner and Shelah proved in [1] that the existence of an (ω,ω_2)-BA is consistent with the axioms of set theory. On the other hand, it is known that under CH there is no (ω,ω_2)-BA and that it is consistent that the continuum is large and ω-thin-very tall Boolean algebras do not exist (see [5] and [9]). Then, our aim here is to prove that it is consistent with the axioms of set theory that there exists an (ω,η)-BA for every $\eta < \omega_3$.

The reader may find in [6] and in the survey paper [10] a wide list of results on superatomic Boolean algebras, as well as a discussion of equivalent definitions and basic facts.

Our set-theoretic terminology is standard and in accordance with [3] or [7].

1. Preliminaries

Let A be a set of ordinals of order type ω_2. We say that a function $F\colon [A]^2 \to [A]^{\leq\omega}$ is a **Δ-function on** A, if $F\{\alpha,\beta\} \subseteq \min\{\alpha,\beta\}$ for all $\alpha,\beta \in A$ with $\alpha \neq \beta$ and for any uncountable set D of finite subsets of A there is an uncountable set $E \subseteq D$ satisfying the following condition:

($\#$) For every $a,b \in E$ with $a \neq b$, $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\tau \in a \cap b$ we have:

(1) $\tau < \alpha,\beta$ implies $\tau \in F\{\alpha,\beta\}$.

(2) $\tau < \beta$ implies $F\{\alpha,\tau\} \subseteq F\{\alpha,\beta\}$.

(3) $\tau < \alpha$ implies $F\{\tau,\beta\} \subseteq F\{\alpha,\beta\}$.

It was proved by Veličković that the existence of a Δ-function on ω_2 follows from $\square_{\omega_1}$ (see [2]). Clearly, a Δ-function on ω_2 can be transferred to a Δ-function on a set of ordinals of order type ω_2.

Baumgartner and Shelah used in [1] a weaker form of a Δ-function in order to construct in a generic extension an (ω,ω_2)-BA.

Then, our aim in this paper is to show the following result:

THEOREM: *Suppose that $\square_{\omega_1}$ holds. Then, there is a c.c.c. partial order that forces the existence of an (ω,η)-BA for every $\eta < \omega_3$.*

So, assume that $\square_{\omega_1}$ holds. Since the countable chain condition is preserved by finite support iterated forcing constructions, in order to find the partial order required in the theorem it is enough to show that for any ordinal $\eta < \omega_3$ there is a c.c.c. partial order $\mathbb{P}_\eta$ that forces the existence of an (ω, η)-BA. To define the partial order $\mathbb{P}_\eta$ we will use the notion of a tree of intervals introduced in [8].

By an **ordinal interval** we mean an interval of the form $[\alpha, \beta)$, where α, β are ordinals with $\alpha < \beta$. Given an ordinal interval $I = [\alpha, \beta)$, we write $I^- = \alpha$ and $I^+ = \beta$.

Let η be a nonzero ordinal. A **tree of intervals** on η is a collection of ordinal intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ where:

(1) $\mathcal{I}_0 = \{[0, \eta)\}$.

(2) For every $I, J \in \mathcal{I}$, $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.

(3) If I, J are different elements of $\mathcal{I}$, $I \subseteq J$ and J^+ is a limit, then $I^+ < J^+$.

(4) $\mathcal{I}_n$ partitions $[0, \eta)$ for each $n < \omega$.

(5) $\mathcal{I}_{n+1}$ refines $\mathcal{I}_n$ for each $n < \omega$.

(6) $\{\alpha\} \in \mathcal{I}$ for every $\alpha < \eta$.

Suppose that $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ is a tree of intervals on an ordinal η. Then:

(1) For each $n < \omega$, we write $E_n = \{ I^- : I \in \mathcal{I}_n \}$.

(2) For $\alpha < \eta$ and $n < \omega$, we write $I(\alpha, n) =$ the interval $I \in \mathcal{I}_n$ such that $\alpha \in I$.

(3) For $\alpha < \eta$, we write $l(\alpha) =$ the least n such that there is an $I \in \mathcal{I}_n$ with $I^- = \alpha$.

Let $\mathcal{I}$ be a tree of intervals on an ordinal η. Let $\alpha < \beta < \eta$ and $I \in \mathcal{I}$. We say that α, β **separate at** I, if for some $n < \omega$, $I = I(\alpha, n) = I(\beta, n)$ and $I(\alpha, n+1) \neq I(\beta, n+1)$. Then, we say that n is the **level where** α, β **separate**, and we write $j(\alpha, \beta) = n$. Note that any pair of different elements of η separate at some interval of $\mathcal{I}$.

Now we say that a tree of intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ is **cofinal**, if the following two conditions hold:

(1) For every $I \in \mathcal{I}_n$ with I^+ a limit ordinal, $E_{n+1} \cap I$ is a sequence of order type $\mathrm{cf}(I^+)$.

(2) For every $I \in \mathcal{I}_n$ with I^+ a successor ordinal, $E_{n+1} \cap I$ is finite.

Then, for every nonzero ordinal η there is a cofinal tree of intervals on η (see [8]).

2. The forcing construction

Assume that $\square_{\omega_1}$ holds. We fix an ordinal η such that $\omega_2 \leq \eta < \omega_3$ and a cofinal tree of intervals $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ on η. For every $n < \omega$ and every $I \in \mathcal{I}_n$ we choose a function F_I: $[E_{n+1} \cap I]^2 \to [E_{n+1} \cap I]^{\leq\omega}$ as follows. If $\mathrm{cf}(I^+) \leq \omega_1$, for every $\alpha, \beta \in E_{n+1} \cap I$ with $\alpha < \beta$ we define $F_I\{\alpha, \beta\} = \{\gamma \in E_{n+1} \cap I : \gamma < \alpha\}$. And if $\mathrm{cf}(I^+) = \omega_2$, F_I is a Δ-function on $E_{n+1} \cap I$. Then, we define the following function F: $[\eta]^2 \to [\eta]^{\leq\omega}$. Suppose $\alpha < \beta < \eta$. Let $I \in \mathcal{I}_n$ be the interval where α, β separate. Let $J, K \in \mathcal{I}_{n+1}$ such that $\alpha \in J$ and $\beta \in K$. Then, we define $F\{\alpha, \beta\} = F_I\{J^-, K^-\} \cup \{I^-\}$.

Next, consider α, β such that $\alpha < \beta < \eta$. Let $k = j(\alpha, \beta)$, $J = I(\alpha, k+1)$ and $K = I(\beta, k+1)$. For $l < \omega$, we write $\beta_l = I(\beta, l)^-$. Then, we define **the walk from α to β via $\mathcal{I}$**, in symbols $w(\alpha, \beta)$, as follows. If $\alpha = J^-$ and $\beta = K^-$, we define $w(\alpha, \beta) = \langle \alpha, \beta \rangle$. If $\alpha \neq J^-$ and $\beta = K^-$, $w(\alpha, \beta) = \langle \alpha, J^+, \beta \rangle$. If $\alpha = J^-$ and $\beta \neq K^-$, then $w(\alpha, \beta) = \langle \alpha, \beta_{k+1}, \beta_{k+2}, \ldots, \beta_{l(\beta)-1}, \beta \rangle$. And if $\alpha \neq J^-$ and $\beta \neq K^-$, $w(\alpha, \beta) = \langle \alpha, J^+, \beta_{k+1}, \beta_{k+2}, \ldots, \beta_{l(\beta)-1}, \beta \rangle$. Note that this notion is in a sense similar to the notion of a walk considered by Todorcevic in [11].

We write $T = \bigcup\{T_\alpha : \alpha < \eta\}$ where $T_\alpha = \{\alpha\} \times \omega$ for each $\alpha < \eta$. T will be the underlying set of the (ω, η)-BA we will construct by forcing. If $s \in T_\alpha$ for some $\alpha < \eta$, we write $\pi(s) = \alpha$ and $l(s) = l(\pi(s))$.

Our c.c.c. notion of forcing will adjoin a partial order $\leq$ on T and a function i: $[T]^2 \to [T]^{<\omega}$ in such a way that for $s, t \in T$ with $s \neq t$, the supremum of $i\{s, t\}$ represents the meet $s \wedge t$.

Then, we define P_η as the set of all $p = (x_p, \leq_p, i_p)$ such that the following conditions hold:

(*) (1) x_p is a finite subset of T.

(2) $\leq_p$ is a partial order on x_p such that for every $s, t \in x_p$, $s <_p t$ implies $\pi(s) < \pi(t)$.

(3) i_p: $[x_p]^2 \to [x_p]^{<\omega}$ satisfying the following:

  (a) If $s <_p t$, then $i_p\{s, t\} = \{s\}$.

  (b) If $s \not<_p t$ and $\pi(s) < \pi(t)$, then $i_p\{s, t\} \subseteq \bigcup\{T_\alpha : \alpha \in F\{\pi(s), \pi(t)\}\}$.

  (c) If $s, t \in x_p$ with $s \neq t$ and $\pi(s) = \pi(t)$, then $i_p\{s, t\} = \emptyset$.

  (d) For all $v \in i_p\{s, t\}$, $v \leq_p s, t$.

  (e) For every $u \leq_p s, t$ there is a $v \in i_p\{s, t\}$ such that $u \leq_p v$.

(4) If $s <_p t$, there are $u_1, \ldots, u_n \in x_p$ with $s <_p u_1 \leq_p \cdots \leq_p u_n \leq_p t$ such that $w(\pi(s), \pi(t)) = \langle \pi(s), \pi(u_1), \ldots, \pi(u_n), \pi(t) \rangle$.

If $p, q \in P_\eta$, we write $p \leq_\eta q$ iff $x_p \supseteq x_q$, $\leq_p \restriction x_q = \leq_q$ and $i_p \restriction [x_q]^2 = i_q$.

We put $\mathbb{P}_\eta = (P_\eta, \leq_\eta)$.

If $\eta = \omega_2$, conditions $(*)(1) - (3)$ correspond to the conditions used by Baumgartner and Shelah in [1]. However, if $\omega_2 < \eta < \omega_3$, we will need condition $(*)(4)$ in order to verify the countable chain condition.

If $p \in P_\eta$ and $s, t \in x_p$ are such that $s \leq_p t$ or $t \leq_p s$, we say that s, t are **comparable in** p.

Suppose that $p \in P_\eta$ and $s <_p t$. Then, if $u_1, \ldots, u_n$ are elements of x_p satisfying $(*)(4)$ for the pair s, t, we say that the sequence $\langle s, u_1, \ldots, u_n, t\rangle$ is **a walk from** s **to** t **in** p. Note that if $\langle v_1, \ldots, v_n\rangle$ is a walk from v_1 to v_n in $p \in P_\eta$, we might have $v_i = v_{i+1}$ for some $i \in \{2, \ldots, n-1\}$. Then, if $\langle v_1, \ldots, v_i, v, \ldots, v, v_j, \ldots, v_n\rangle$ is a walk from v_1 to v_n in p, we shall also consider the sequence $\langle v_1, \ldots, v_i, v, v_j, \ldots, v_n\rangle$ as a walk from v_1 to v_n in p.

LEMMA 1: *If $\mathbb{P}_\eta$ preserves cardinals, forcing with $\mathbb{P}_\eta$ adjoins an (ω, η)-BA.*

Proof: Let G be a $\mathbb{P}_\eta$-generic filter. We write $\leq = \bigcup\{\leq_p : p \in G\}$ and $i = \bigcup\{i_p : p \in G\}$. It is easy to check that $T = \bigcup\{x_p : p \in G\}$ and that $\leq$ is a partial order on T.

Now suppose that $\alpha < \beta < \eta$ and $t \in T_\beta$. We prove that the set $\{s \in T_\alpha : s < t\}$ is infinite. To check this point, suppose that $p \in P_\eta$ and $n_0 < \omega$. We show that there is a $q \in P_\eta$ such that $q \leq_\eta p$ and $(\alpha, n) <_q t$ for some $n > n_0$. We may assume that $t \in x_p$. Consider $n > n_0$ such that for every $u \in x_p$, if $u = (\pi(u), m)$ then $m < n$. Let $k = j(\alpha, \beta)$. We assume $l(\alpha) = k+1$ and $l(\beta) > k+1$. Otherwise, the considerations are similar. We put $s = (\alpha, n)$, and for $k < l < l(\beta)$ we write $t_l = (\beta_l, n)$ where $\beta_l = I(\beta, l)^-$. Then, we define $q = (x_q, \leq_q, i_q)$ as follows:

(1) $x_q = x_p \cup \{s\} \cup \{t_l : k < l < l(\beta)\}$.

(2) $<_q = <_p \cup \{(s, v) : t \leq_p v\} \cup \{(t_l, v) : t \leq_p v \text{ and } k < l < l(\beta)\} \cup \{(s, t_l) : k < l < l(\beta)\} \cup \{(t_l, t_m) : k < l < m < l(\beta), t_l \neq t_m\}$.

(3) $i_q\{u, v\} = i_p\{u, v\}$ if $u, v \in x_p$; $i_q\{u, v\} = u$ if $u <_q v$; $i_q\{u, v\} = v$ if $v <_q u$; and $i_q\{u, v\} = \emptyset$ otherwise.

We show that $q \in P_\eta$. The verification of $(*)(1) - (3)$ is straightforward. Then, we check $(*)(4)$. Suppose that $u <_q v$. If $u <_p v$, we are done. So, we assume $u \not<_p v$. It follows from the definition of $<_q$ that $u \in \{s, t_{k+1}, \ldots, t_{l(\beta)-1}\}$ and $v \in \{t_{k+1}, \ldots, t_{l(\beta)-1}, t\} \cup \{v' \in x_p : t <_p v'\}$. Suppose that $v = t_m$ for $k < m < l(\beta)$. Then, if $u = s$ we have that $\langle s, t_{k+1}, \ldots, t_m\rangle$ is a walk from s to t_m in q; and if $u = t_l$ for $k < l < m$, then $\langle t_l, t_{l+1}, \ldots, t_m\rangle$ is a walk from t_l to t_m in q. If $v = t$, we proceed in a similar way.

Now, we suppose $u = s$ and $t <_p v$. If $u \in \{t_l : k < l < l(\beta)\}$ and $t <_p v$, the considerations are analogous. Let $\gamma = \pi(v)$ and $m = j(\alpha, \gamma)$. Since $j(\alpha, \gamma) = m$, we have $l(\gamma) \geq m+1$. We assume $l(\gamma) > m+1$ (the situation $l(\gamma) = m+1$ is similar and easier to handle). Note that since $\alpha < \beta < \gamma$, we have $m \leq k$. Then, we distinguish the following cases:

CASE 1: $m < k$.
It follows that $\beta \in I(\alpha, m+1)$, and therefore $j(\beta,\gamma) = m$. Let

$$\langle t, t', v_{m+1}, \ldots, v_{l(\gamma)-1}, v \rangle$$

be a walk from t to v in p. Then, $\langle s, t', v_{m+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from s to v in q.

CASE 2: $m = k$ and $\gamma \notin I(\beta, k+1)$.
Consider a walk $\langle t, t', v_{k+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ from t to v in p. Then,

$$\langle s, v_{k+1}, \ldots, v_{l(\gamma)-1}, v \rangle$$

is a walk from s to v in q.

CASE 3: $m = k$ and $\gamma \in I(\beta, k+1) \setminus I(\beta, l(\beta))$.
Let $l = j(\beta,\gamma)$. We have $l(t), l(v) \geq l+1$. First, assume $l(t), l(v) > l+1$. Then, if $\langle t, t', v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from t to v in p, we infer that $\langle s, t_{k+1}, \ldots, t_l, v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from s to v in q. Analogously, if $l(t) = l+1$ and $l(v) > l+1$, we have that if $\langle t, v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from t to v in p, then $\langle s, t_{k+1}, \ldots, t_l, v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from s to v in q. And if $l(v) = l+1$, it is clear that $\langle s, t_{k+1}, \ldots, t_l, v \rangle$ is a walk from s to v in q.

CASE 4: $m = k$ and $\gamma \in I(\beta, l(\beta))$.
Let $l = j(\beta,\gamma)$. Assume $l(v) > l+1$. Then, if $\langle t, v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from t to v in p, we have that $\langle s, t_{k+1}, \ldots, t_{l(\beta)-1}, t, v_{l+1}, \ldots, v_{l(\gamma)-1}, v \rangle$ is a walk from s to v in q. On the other hand, if $l(v) = l+1$ note that $\langle t, v \rangle$ is the only walk from t to v in p, and then we obtain that $\langle s, t_{k+1}, \ldots, t_{l(\beta)-1}, t, v \rangle$ is a walk from s to v in q.

Thus, proceeding as in [1, Theorem 7.1] we can show that $\mathbb{P}_\eta$ adjoins an (ω, η)-BA. ∎

Now, our purpose is to prove the following result:

LEMMA 2: *$\mathbb{P}_\eta$ is c.c.c.*

We need to introduce some definitions in order to prove Lemma 2.

Let $f \colon y \to z$ be a bijection, where y, z are finite subsets of T. We say that f is **adequate**, if the following two conditions hold:

(1) For every $s,t \in y$, $\pi(s) < \pi(t)$ iff $\pi(f(s)) < \pi(f(t))$.

(2) For all $s = (\alpha, n) \in y$, $f(\alpha, n) = (\beta, m)$ implies $n = m$.

Now, we say that a set $Z \subseteq P_\eta$ is **separated**, if the following conditions hold:

(1) $\{x_p: p \in Z\}$ forms a Δ-system with root x.

(2) For each $\alpha < \eta$, either $x_p \cap T_\alpha = x \cap T_\alpha$ for every $p \in Z$, or there is at most one $p \in Z$ such that $x_p \cap T_\alpha \neq \emptyset$.

(3) For every $p, q \in Z$ there is an adequate bijection $h_{pq}: x_p \to x_q$ such that:
   (a) For any $s \in x$, $h_{pq}(s) = s$.
   (b) For all $s,t \in x_p$, $s \leq_p t$ iff $h_{pq}(s) \leq_q h_{pq}(t)$.
   (c) For all $s,t \in x_p$ with $s \neq t$, $h_{pq}[i_p\{s,t\}] = i_q\{h_{pq}(s), h_{pq}(t)\}$.
   (d) For every $s \in x_p$, $l(s) = l(h_{pq}(s))$.

Note that if $Y, Z \subseteq P_\eta$ with $Y \subseteq Z$ and Z is separated, then Y is also separated.

If Z is a separated subset of P_η, we denote by $x(Z)$ the root of the set $\{x_p$: $p \in Z\}$. It should be noted that if Z is separated, $x = x(Z)$ and $a = \pi[x]$, then for all $p \in Z$ and $u \in x_p$, we have $u \in x$ iff $\pi(u) \in a$.

The following lemma can be proved by means of a standard combinatorial argument.

LEMMA 3: *Every uncountable subset of P_η has an uncountable separated subset.*

The following two basic properties of separated sets will be essential in the sequel.

LEMMA 4: *Let Z be a separated subset of P_η. Let $x = x(Z)$. Suppose that $p, q \in Z$ and $u, s \in x_p$ with $u \in x$, $s \in x_p \setminus x$ and $u <_p s$. Let $s' = h_{pq}(s)$. Suppose that $I \in \mathcal{I}$ is such that $\pi(u), \pi(s) \in I$. Then, $\pi(s') \in I$.*

Proof: Suppose that $I \in \mathcal{I}_n$ is an ordinal interval containing $\pi(u)$ and $\pi(s)$. Assume that $\pi(s') \notin I$. Let $m = j(\pi(u), \pi(s'))$. Clearly, $m < n$. Let $J = I(\pi(s'), m+1)$. Since $u <_q s'$, by using $(*)(4)$ we deduce that there is a $v' \in x_q \cap T_{J^-}$ such that $u <_q v' \leq_q s'$. As $v' \in T_{J^-}$, we have $l(v') \leq n$. Now, consider $v = h_{qp}(v')$. Since Z is separated, we have $u <_p v \leq_p s$ and $l(v) \leq n$, which contradicts the fact that $u, s \in I \in \mathcal{I}_n$. ∎

LEMMA 5: *Let Z be an uncountable separated subset of P_η. Let $x = x(Z)$. Then, $i_p \restriction [x]^2 = i_q \restriction [x]^2$ for all $p, q \in Z$.*

Proof: It is enough to show that $i_p\{s,t\} \subseteq x$ for all $p \in Z$ and $s, t \in x$. Suppose on the contrary that there are $p \in Z$ and $s, t \in x$ with $i_p\{s,t\} \not\subseteq x$. We immediately deduce from $(*)(3)(a)$ that s, t are not comparable in p. Now, by using the fact that Z is separated, we infer that $i_p\{s,t\} \not\subseteq x$ for all $p \in Z$. Then since Z is uncountable, we deduce from $(*)(3)(b)$ that the set $F\{\pi(s), \pi(t)\}$ is also uncountable, which contradicts the fact that $F\colon [\eta]^2 \to [\eta]^{\leq\omega}$. ∎

If Z is a separated subset of P_η, we write $a_p = \pi[x_p]$ for every $p \in Z$.

Let Z be an uncountable separated subset of P_η. We say that Z is **admissible** if for every $I \in \mathcal{I}_n$ such that $|E_{n+1} \cap I| = \omega_2$ and $\{a_p \cap E_{n+1} \cap I\colon p \in Z\}$ is uncountable, we have that $\{a_p \cap E_{n+1} \cap I\colon p \in Z\}$ satisfies condition (#) for the function F_I.

Lemma 6: *Every uncountable separated subset of P_η has an admissible subset.*

Proof: Let Z be an uncountable separated subset of P_η. Let $x = x(Z)$ and $a = \pi[x]$. Let $k = |a_p \setminus a|$ for $p \in Z$. For $i \in \{1, \ldots, k\}$, we denote by $a_p(i)$ the i-element in the order of $a_p \setminus a$. Then, we define a sequence $Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_k$ of uncountable subsets of Z as follows. We put $Z_0 = Z$. Consider $i \in \{1, \ldots, k\}$. First, we choose an $n < \omega$ and an interval $K_i \in \mathcal{I}_n$ such that $\{a_p(i) \in K_i\colon p \in Z_{i-1}\}$ is uncountable but for every $J \in \mathcal{I}$ with $J \subsetneqq K_i$, $\{a_p(i) \in J\colon p \in Z_{i-1}\}$ is countable. Note that such an interval K_i exists, because otherwise we would find an infinite strictly decreasing sequence of ordinals in η. Now, we put $Z_i' = \{p \in Z_{i-1}\colon a_p(i) \in K_i\}$. Then, if $|E_{n+1} \cap K_i| = \omega_2$ and $\{a_p(i)\colon p \in Z_i'\} \subseteq E_{n+1} \cap K_i$, we choose an uncountable subset Z_i of Z_i' such that $\{a_p \cap E_{n+1} \cap K_i\colon p \in Z_i\}$ satisfies (#) for the function F_{K_i}. Otherwise, we put $Z_i = Z_i'$.

We define $Y = Z_k$. Consider $I \in \mathcal{I}_n$ such that $|E_{n+1} \cap I| = \omega_2$ and $\{a_p \cap E_{n+1} \cap I : p \in Y\}$ is uncountable. It follows that for some $i \leq k$, $\{a_p(i) \in E_{n+1} \cap I\colon p \in Y\}$ is uncountable. Then, by the construction of Y, we deduce that $I = K_i$ and $\{a_p(i) : p \in Z_i'\} \subseteq E_{n+1} \cap I$. So, Y is as required. ∎

Now, our aim is to prove the following result:

Lemma 7: *Every admissible subset of P_η is linked.*

Proof: Let Z be an admissible subset of P_η. Let $x = x(Z)$. As above, we write $a_p = \pi[x_p]$ for $p \in Z$ and $a = \pi[x]$. Consider $p, q \in Z$ with $p \neq q$. Our purpose is to show that p, q are compatible. It follows from Lemma 5 that $i_p \restriction [x]^2 = i_q \restriction [x]^2$. We define $r = (x_r, \leq_r, i_r)$ as follows. We put $x_r = x_p \cup x_q$. If $s, t \in x_r$, we set $s \leq_r t$ iff $s \leq_p t$ or $s \leq_q t$ or there is a $u \in x$ such that either $s \leq_p u \leq_q t$ or $s \leq_q u \leq_p t$. Now, for $s, t \in x_r$ with $s \neq t$ we define

$i_r\{s,t\}$ as follows: $i_r\{s,t\} = i_p\{s,t\}$ if $s,t \in x_p$; $i_r\{s,t\} = i_q\{s,t\}$ if $s,t \in x_q$; and if $s \in x_p \setminus x$ and $t \in x_q \setminus x$, we define $i_r\{s,t\} = \{u \in x_r\colon u \leq_r s,t$ and $\pi(u) \in F\{\pi(s),\pi(t)\}\}$ if s,t are not comparable in r, $i_r\{s,t\} = \{s\}$ if $s <_r t$ and $i_r\{s,t\} = \{t\}$ if $t <_r s$.

We show that $r \in P_\eta$. It is easy to check $(*)(1)-(2)$ and $(*)(3)$(a)–(d). Then, we prove $(*)(3)$(e). To check this point, consider $s,t \in x_r$ with $s \neq t$. First, suppose that $s,t \in x_p$. Consider $u \in x_r$ such that $u \leq_r s,t$. If $u \in x_p$, we are done. Then, assume that $u \in x_q \setminus x$. Let $w_1, w_2 \in x$ such that $u \leq_q w_1 \leq_p s$ and $u \leq_q w_2 \leq_p t$. Let $w \in i_q\{w_1,w_2\}$ such that $u \leq_q w$. Note that $w \in x$. Now there is a $v \in i_p\{s,t\}$ such that $w \leq_p v$, and hence $u \leq_r v$. If $s,t \in x_q$, the argument is parallel.

Now, suppose that $s \in x_p \setminus x$ and $t \in x_q \setminus x$ are not comparable in r. Suppose $\pi(s) < \pi(t)$. Consider $u \in x_r$ with $u <_r s,t$. Let $I \in \mathcal{I}_k$ be the interval where $\pi(s), \pi(t)$ separate. Let $J = I(\pi(s), k+1)$ and $K = I(\pi(t), k+1)$. We distinguish the following cases:

Case 1: $\pi(u) \in a \setminus I$.

Note that $u <_r s$ implies $u <_p s$, and $u <_r t$ implies $u <_q t$. Since $u <_p s$, we infer by $(*)(4)$ that there is a $v_1 \in x_p \cap T_{I^-}$ such that $u <_p v_1 \leq_p s$. Analogously, there is a $v_2 \in x_q \cap T_{I^-}$ such that $u <_q v_2 <_q t$. Therefore, $I^- \in a$. So, we have $\pi(s) \neq I^-$. Now, by using $(*)(3)$(c), we deduce that $v_1 = v_2$. Thus, since $I^- \in F\{\pi(s),\pi(t)\}$, we are done.

The cases $\pi(u) \in a_p \setminus (a \cup I)$ and $\pi(u) \in a_q \setminus (a \cup I)$ are similar to Case 1.

Case 2: $\pi(u) \in a \cap I$.

Note that if $\pi(u) = I^-$, as $I^- \in F\{\pi(s),\pi(t)\}$, we are done. So, we assume $\pi(u) \neq I^-$.

First, we prove that $\pi(u) \notin J$. To show this, assume on the contrary that $\pi(u) \in J$. Let $s' = h_{pq}(s)$ and $t' = h_{qp}(t)$. We infer from Lemma 4 that $\pi(s') \in J$ and $\pi(t') \in I \setminus J$. Let $K_0 = I(\pi(t'), k+1)$. Note that since $\pi(t') \notin J$, we have $J \neq K_0$. Suppose that $s \not<_p t'$. As $u <_p s,t'$, there is a $v \in i_p\{s,t'\}$ with $u \leq_p v$. Now since $s \not<_p t'$ and $\pi(s) < \pi(t')$, we infer from $(*)(3)$(b) that $\pi(v) \in F\{\pi(s),\pi(t')\} = F_I\{J^-, K_0^-\} \cup \{I^-\}$. Then since $F_I\{J^-, K_0^-\} \subseteq J^-$, $\pi(u) \in J$ and $\pi(u) > I^-$, we deduce that $\pi(v) < \pi(u)$, which contradicts the fact that $u \leq_p v$. Thus, we have $s <_p t'$. Then, by using $(*)(4)$, we infer that there is a $u_1 \in x_p \cap T_{J^+}$ with $s <_p u_1 \leq_p t'$. Also, as $s' <_q t$ and $\pi(s') \in J$, we infer that there is a $u_2 \in x_q \cap T_{J^+}$ such that $s' <_q u_2 \leq_q t$. Hence, $J^+ \in a$ and $u_1, u_2 \in x$. But since $u <_p u_1, u_2$, we infer from $(*)(3)$(c) that $u_1 = u_2$, and so $s <_r t$, which contradicts the assumption that s,t are not comparable in r. This completes the proof that $\pi(u) \notin J$.

Then, by using again $(*)(4)$, there are $v_1 \in x_p \cap T_{J^-}$ and $v_2 \in x_q \cap T_{K^-}$ such that $u <_p v_1 \leq_p s$ and $u <_q v_2 \leq_q t$. Our purpose is to verify that there is a $v \in x_r$ such that $u \leq_r v <_r s, t$ and $\pi(v) \in F\{\pi(s), \pi(t)\}$. To this end, we distinguish the following three situations: (a) $v_1 \in x$; (b) $v_2 \in x$; and (c) $v_1, v_2 \notin x$. First, suppose that $v_1 \in x$. Then since $\pi(v_1) \in J$ and $v_1 <_p s$, by the argument given in the preceding paragraph, we infer that $v_1 \not\leq_q t$. Now as $u <_q v_1, t$, there is a $v \in i_q\{v_1, t\}$ such that $u \leq_q v$. So, we have $v <_q v_1 <_p s$ and $v <_q t$, and thus $u \leq_r v <_r s, t$. But since $v_1 \not\leq_q t$ and $\pi(v_1) < \pi(t)$, we deduce from $(*)(3)(b)$ that $\pi(v) \in F\{\pi(v_1), \pi(t)\} = F\{\pi(s), \pi(t)\}$, and so we are done. On the other hand, note that if $v_2 \in x$, we would have $u <_p s, v_2$ and s, v_2 incomparable in p, and then since $I(\pi(v_2), k+1) = I(\pi(t), k+1)$ we can proceed in a similar way.

Now, we suppose that $v_1, v_2 \notin x$. Let $J_0 = I(\pi(u), k+1)$. Note that if $\pi(u) \neq J_0^-$, then since $u <_p v_1$ and $u <_q v_2$, by using $(*)(4)$ and $(*)(3)(c)$, we infer that there is a $w \in x \cap T_{J_0^+}$ such that $u <_p w$, $w <_p v_1$ and $w <_q v_2$. So, we may assume that $\pi(u) \in E_{k+1} \cap I$. We want to show that $\pi(u) \in F\{\pi(s), \pi(t)\}$. We deduce from Lemma 4 that $\pi(h_{pp'}(v_1)) \in I$ for every $p' \in Z$, and hence $\{a_p \cap E_{k+1} \cap I : p \in Z\}$ is uncountable. Without loss of generality, we may suppose $|E_{k+1} \cap I| = \omega_2$. Then, as Z is admissible, $\{a_p \cap E_{k+1} \cap I : p \in Z\}$ satisfies (#) for the function F_I. We have $\pi(u), \pi(v_1), \pi(v_2) \in E_{k+1} \cap I$ and $\pi(u) < \pi(v_1) < \pi(v_2)$. Since $v_1, v_2 \notin x$, we infer that $J^- \in a_p \setminus a$, $K^- \in a_q \setminus a$. Then as $\pi(u) \in a$, we deduce that $\pi(u) \in F_I\{J^-, K^-\} \subseteq F\{J^-, K^-\} = F\{\pi(s), \pi(t)\}$.

Case 3: $\pi(u) \in (a_p \setminus a) \cap I$.

We assume $\pi(u) \neq I^-$. Consider $w \in x$ such that $u <_p w <_q t$. Since s, t are not comparable in r, it follows that $s \not<_p w$. Also, if $w <_p s$ we would apply Case 2. So, we suppose that s, w are incomparable in p.

Note that if $\pi(w) \in K$, then since $u <_p s, w$ there would exist a $v \in i_p\{s, w\}$ such that $u \leq_p v$ and $\pi(v) \in F\{\pi(s), \pi(w)\} = F\{\pi(s), \pi(t)\}$. So, as $v <_r s, t$, we are done. Thus, assume that $\pi(w) \notin K$. Since $w <_q t$ and $w <_p h_{qp}(t)$, by using $(*)(4)$ and $(*)(3)(c)$, we may suppose that $\pi(w) \in E_{k+1} \cap I$. Since s, w are incomparable in p, $u <_p s, w$ and $\pi(s) \in J$, we deduce from $(*)(3)(e)$ and $(*)(3)(b)$ that $\pi(u) \notin J$. Then, by $(*)(4)$, there are $v_1 \in x_p \cap T_{J^-}$ such that $u <_p v_1 \leq_p s$ and $v_2 \in x_q \cap T_{K^-}$ such that $w <_q v_2 \leq_q t$. Suppose that $v_1 \in x$. As $u <_p v_1, w$, there is a $v \in i_p\{v_1, w\}$ such that $u \leq_p v$. But since $v \in x$ and $v <_r s, t$, by Case 2 we are done. Also, if $v_2 \in x$ we would have $u <_p s, v_2$ and s, v_2 incomparable in p, and then since $I(\pi(v_2), k+1) = I(\pi(t), k+1)$ it is easy to obtain the desired conclusion.

Therefore, we may suppose that $v_1 \in x_p \setminus x$ and $v_2 \in x_q \setminus x$. Note that if $v_1 <_p w$, we have $v_1 <_p s, w$, and thus since s, w are not comparable in p, we deduce from $(*)(3)(e)$ and $(*)(3)(b)$ that $\pi(v_1) < J^-$, which contradicts the fact that $v_1 \in T_{J^-}$. On the other hand, if $w \leq_p v_1$, we deduce that $w <_r s, t$, and therefore we could apply again Case 2. So, we suppose that v_1, w are incomparable in p. Then as $u <_p v_1, w$, there is a $v \in i_p\{v_1, w\}$ such that $u \leq_p v$ and $\pi(v) \in F\{\pi(v_1), \pi(w)\}$. Since $v <_p v_1, w$, it is clear that $v <_r s, t$. We want to show that $\pi(v) \in F\{\pi(s), \pi(t)\}$. Note that as $w <_q v_2$ and $l(v_2) = k+1$, we infer from Lemma 4 that $\{a_p \cap E_{k+1} \cap I : p \in Z\}$ is uncountable. Without loss of generality, we may assume $|E_{k+1} \cap I| = \omega_2$. Since Z is admissible, $\{a_p \cap E_{k+1} \cap I : p \in Z\}$ satisfies (#) for the function F_I. Then, since $\pi(w) < \pi(v_2)$, it is easy to check that $F\{\pi(v_1), \pi(w)\} \subseteq F\{\pi(v_1), \pi(v_2)\}$. So we have $\pi(v) \in F\{\pi(v_1), \pi(w)\} \subseteq F\{\pi(v_1), \pi(v_2)\} = F\{\pi(s), \pi(t)\}$.

The case $\pi(u) \in (a_q \setminus a) \cap I$ is similar to Case 3.

Next, we prove that $(*)(4)$ holds for r. Consider $s, t \in x_r$ such that $s <_r t$. If $s, t \in x_p$ or $s, t \in x_q$, we are done. Assume that $s \in x_p \setminus x$ and $t \in x_q \setminus x$. Let $u \in x$ such that $s <_p u <_q t$. Let $k = j(\pi(s), \pi(t))$, $\alpha = \pi(s)$, $\beta = \pi(t)$ and $\gamma = \pi(u)$. Put $J = I(\pi(s), k+1)$ and $K = I(\pi(t), k+1)$. Without loss of generality, we may assume $\pi(s) > J^-$ and $\pi(t) > K^-$. Then, we distinguish the following cases:

Case 1: $\pi(u) \in J$.

It follows that $k = j(\pi(u), \pi(t))$. Then, if $\langle u, u', t'_{k+1}, \ldots, t'_{l(\beta)-1}, t\rangle$ is a walk from u to t in q, we have that $\langle s, u', t'_{k+1}, \ldots, t'_{l(\beta)-1}, t\rangle$ is a walk from s to t in r.

Case 2: $\pi(u) \in K$.

Put $m = j(\pi(u), \pi(t))$. Note that $k = j(\pi(s), \pi(u))$. Suppose that

$$\langle s, s', u'_{k+1}, \ldots, u'_{l(\gamma)-1}, u\rangle$$

is a walk from s to u in p and $\langle u, \bar{u}, \bar{t}_{m+1}, \ldots, \bar{t}_{l(\beta)-1}, t\rangle$ is a walk from u to t in q. Then, $\langle s, s', u'_{k+1}, \ldots, u'_m, \bar{t}_{m+1}, \ldots, \bar{t}_{l(\beta)-1}, t\rangle$ is a walk from s to t in r.

Case 3: $\pi(u) \notin J \cup K$.

Note that $j(\pi(s), \pi(u)) = j(\pi(u), \pi(t)) = k$. Since $s <_p u$, there is an $s' \in x_p \cap T_{J^+}$ such that $s <_p s' \leq_p u$. Now suppose that $\langle u, u', t'_{k+1}, \ldots, t'_{l(\beta)-1}, t\rangle$ is a walk from u to t in q. Then, $\langle s, s', t'_{k+1}, \ldots, t'_{l(\beta)-1}, t\rangle$ is a walk from s to t in r.

This completes the proof that $r \in P_\eta$. Then, it is clear that $r \leq_\eta p, q$. ∎

So, we have proved Lemma 2, and thus we obtain our main result. Also, note that as a consequence of our forcing construction, we obtain the following consistency result:

COROLLARY: *$MA + 2^\omega > \omega_2 + \square_{\omega_1}$ implies the existence of an (ω, α)-BA for every $\alpha < \omega_3$.*

So, if we consider the forcing of Solovay and Tennenbaum over a ground model of $V = L$, we obtain that in the generic extension there is an (ω, α)-BA for every $\alpha < \omega_3$.

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